Journal of Business; ISSN: 2233-369X ; e-ISSN: 2346-8297 ; Volume 7, Issue 1, 2018

Theoretical Foundations of Multi-Product Prices Optimization Model (Part1)

Tamta LEKISHVILI *

Abstract

The paper investigates and analyses the technologies and the methods for price optimization in a multi-product environment, where the firms face more difficulties to make an optimal decisions. Multi-Product Prices Optimization (MPPO) problem emerges specific problems, because of both prices and demands, being random variables, may exhibit different covariate relationships, determined by means of substitutional and complimentarity effects. The system of covariate relationships, determined by means of substitution and complimentarity processes, is represented as the system of multivariance regression equations. MPPO problem have been reduced to the maximization problem of the objective function of Total Revenue, which is a quadratic form. In multidimensional cases one faces the problem of absence of local maximum of Total Revenue function. The simple criterion of existing the optimal solution of the MPPO problem is suggested. In the case of absence of the strict maximum of Total Revenue function, parameters should be calculated on the base of OPPO model, that is on the base of independent paired problems. The represented theory can be used as a basis for development of a relevant software.

Keywords: complementarity, multi-product pricing, price optimization, single-product pricing, substitution, total revenue
JEL: M21

Introduction

Problem Statement

Correct and timely estimation of economic, financial and managerial parameters of an enterprise, such as: maximum of total revenue and profit, price and cross elasticities of demand, marginal revenues of the complex of products under condition of cross-elasticity correlations is one of the most demanded problem in theory and practice of economics and business. In case of single product, well known linear relationship between price and demand is expressed by

$$D=a+kP+\varepsilon, \quad (1)$$

where k-slope, and, it has the same dimension as D, that is, number of units sold per unit of time;

a-intercept (a>0), again the same dimension as D, but let us note that it has no direct economic sense;

ε-normally distributed random variable with zero mean.

which can be solved using standard least squares methods. This model gives us possibility to estimate regression coefficients correctly and based on this to identify the above mentioned managerial parameters.

In contrast to a single-product pricing models, multi-product pricing models have been significantly less studied due to the complexity of multi-product demand function (Soon, Zhao, & Zhang, 2009, pp. 399-430). Many decision-makers use incomplete demand functions which are defined only on a limited scale, e.g. the combination where all elements of demand functions are non-negative. In real marketplace managers often make pricing decisions for several products simultaneously. By doing so previous paragraph, decision makers can control for substitution effects or benefit from potential synergies between products (Goic, 2011, p. 547).

Companies sell several products, and in addition to determining the quantity of orders should also determine the selling price of each product sold, whereas the main component of determining the price - the demand, for each product is conditional and depends on the demand of other products.

* Ph.D-c., Faculty of Business Management, International Black Sea University, Tbilisi, Georgia. Email:tlekishvili@ibsu.edu.ge

Based on the defined problem in the previous paragraph, our objectives and research goals are to create mathematical model and corresponding software that can be applied in solving particular pricing problems. The optimization methods for one product are well known, but they cannot be used in the multi-product case, because of an effect of a cross-elasticity interaction among products (Gallego & Wang, 2014, pp. 450-461) . The latter leads to complicate mutual correlations among products prices and demands, and therefore requires usage of alternative mathematical methods. Developing methods and corresponding algorithms for solution of the defined problem allows, based on the accumulated statistical data of the prices and demands of the products, obtaining comprehensive analysis of economic, financial and managerial states of the enterprise. Specifically, the results of the research allow estimating: maximum of total revenue and profit, price and cross elasticity of demand, marginal revenues of the complex of products, break even points. It should be mentioned that there is no Business Plan without the analysis of revenue, profit, break even points; therefore, the research goal can be also crucially important for easing the business processes.

Methodology

Deductive approach is adopted making maximum use of existing theory, to establish conceptual framework and test the adequacy of the framework by applying empirical facts. In order to estimate the methods of price optimization the market situation need to be assessed with some criteria, like profit dynamics, consumer loyalty and so on. To capture qualitative data various measures are to be utilized.

Qualitative as well as quantitative approach of data gathering and analysis is used, including an analysis of official documents, publications and articles. Primary data is sought through semi-structure qualitative interviews from several retailers and consumers for in-depth insights. Some quantitative data gathering and analysis might be considered too. The findings will be limited to particular time and case.

The further research would make a contribution to the field, and could be tackled in a two year project. This study will fill the growing gap between the theory and the practice. The findings from this case are to play important role in general theoretical discussion. It is to scrutinize and challenge existing, well-established theoretical assumptions and provide new insights. For retailers it will provide scientifically affirmed account to what went wrong in their business and what can be improved.

Methodology consists in usage of the methods of mathematical statistics, specifically regression analysis and hypothesis testing ideology, optimization methods of quadratic programming, matrices method of linear algebra and programming in Matlab language.

One-Product Prices Optimization (OPPO) Problem

The OPPO Problem is well known. It is based on the assumption that the dependence between Demand, D, and Price, P follows equation (1).

The one of the basic lows of economics reveals that k<0, so that the increasing of Price implies strictly decreasing of Demand. It is clear that this is an idealization of real economic processes, but it is a direct consequence of the assumption (1), which also implies that Total Revenue can be represented as

$$TR=aP+kP^2 \quad (2)$$

It is easy to obtain values of prices and demand which provide maximum of Total Revenue. We omit well known details and directly represent the necessary expressions

$$P_{op}=\frac{a}{2k}; \quad (3)$$

$$D_{op}=\frac{a}{2}; \quad (4)$$

$$Tr_{max}=\frac{a^2}{4k}. \quad (5)$$

The represented optimal values don't depend on costs, which also can be represented by means of straight line equation

$$C_T=C_F+c_vD, \quad (6)$$

where C_T - total costs, C_F- fixed costs, and c_v – variable costs.

Usage of the costs requires inverting of (1)

$P=a_1- k_1D$, (1')

where $a_1=a/k$; $k_1=1/k$.

Now Total Revenue can be expressed in terms of Demand,

$$TR=a_1D+k_1D^2,$$

and one can use (6) to calculate Profit function and points D_1 and D_2 where Profit is equal to costs, that is Break-even points

$$Profit=TR-C= a_1D-k_1D^2- C_F-c_vD=0;$$

$$D_{1,2}=\frac{-(c_v-a_1)\pm\sqrt{(c_v-a_1)^2-4*C_F*k_1}}{2k_1}. \quad (7)$$

Also, it is easy to calculate parameters corresponding to (3)÷(5)

$$P_{op}^c=\frac{a_1+c_v}{2}; \quad (3')$$

$$D_{op}^c=\frac{a_1-c_v}{2k_1}; \quad (4')$$

$$Prmax=\frac{a_1^2-c_v^2}{4k_1}. \quad (5')$$

All these parameters are extremely useful for imple-

Journal of Business; ISSN: 2233-369X ; e-ISSN: 2346-8297 ; Volume 7, Issue 1, 2018

mentation of efficient financial and economic management in a retail organization activity. Note, that they are strongly depended on correct and reliable estimation of regression parameters a, k and ε in (1). Despite of the usefulness of the model, it cannot be applied when a manager faces Multi-Product Prices Optimization (MPPO) problem. Besides that the latter Problem differs from OPPO problem with high dimensions of all of the functions represented above, it has another specific issue: both prices and demands, being random variables, may exhibit different covariate relationships, determined by means of substitution and complimentarity processes. Such kind of covariate relationships creates a system of constraints which should be included in the relevant mathematical model of the MPPO problem.

Development of the Mathematical Model for the MPPO Problem

Determination of the Model

The basic objects of the model are observed random variables: di (i=1, 2,…,m) - demand and pi (i=1,2,…,m)-corresponding prices. We don't consider question about their distributions. They constitute two m-dimensional vectors d (d1,…,dm) and p(p1,…, pm). We assume that between their components m relationships of type (1) exist

$$d_i=a_i+k_{ii}p_i+\varepsilon_i \text{ (i=1, 2,…,m)}, \quad (8)$$

where all parameters have the same sense as in (1). We call (8) paired models and their parameters - paired parameters. Note that paired model is nothing but the set of m independent OPPO models.

Total Revenue function TR_M of the MPPO problem can be represented as an inner product of two m-dimensional vectors d and p

$$TR_M=(d,p), \quad (9)$$

or, in coordinate form

$$TR_M=\sum_{i=1}^{m} d_i p_i. \quad (10)$$

Drastic distinction between paired models and MPPO models is that each of demand potentially can be dependent on all/part of price variables

$$\begin{aligned} d_1 &= \gamma_1\ p_1+...+\gamma_{1m}p_m+b_1+\varepsilon_1 \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ d_m &= \gamma_{m1}p_1+...+\gamma_{\boldsymbol{m}}\ \ p_m+b_m+\varepsilon_m \end{aligned} \quad (11)$$

Where

ε-normally distributed random variable with zero mean.

Observe, that γ_{ij} (i, j=1,2,…,m) now may have positive or negative signs, as demands can be in different relations (substitute or complimentary) with different prices.

(11) can be represented in more economic, matrix, format

$$d=\Gamma p+b+\varepsilon, \quad (12)$$

where d and p - m×1 vectors of demand and prices;

b- m×1 vector of intercepts;

Γ – m-order matrix, with entries γij equal to coefficients in (11) and subject to identification.

ε - random m×1 vector, contained normally distributed random variables with zero mean and covariance matrix Σ , which completely defines statistical nature of the system of equations in (12), and therefore their identification method. This question will be discussed later in detail.

If the entries γij of the matrix Γ have already being identified, then substituting of (12) into (10) implies (after simple transformations)

$$TR_M=p^TKp+(p,b), \quad (13)$$

where *K*-simmetric matrix with entries

$$k_{ij}=\begin{cases}\gamma_{ij} & if\ i=j\\ \frac{(\gamma_{ij}+\gamma_{ji})}{2} & if\ i\neq j\end{cases} \text{ (i=1, 2,…,m).}$$

Considering natural constraints of non-negativity of prices,

$$0\le p_i \text{ (i=1, 2,…,m)} \quad (14)$$

the MPPO problem have been reduced to the maximization problem of the objective function (13) (which is a quadratic form with symmetric matrix K) with constraint (14). Thus, the MPPO problem can be solved by means of well-known quadratic programming methodology (Best, 2017).

Results

Existence of the Solution and Results

Quadratic forms have one critical point, which may be one out of three types: maximum, minimum or saddle points. Type of the points depends on signs of entries of the matrix K. In one dimensional case we have guaranteed maximum, because of negative sign of slope k and positive sign of intercept a in (1), which are consequences of economic lows. There is different situation in multi-dimensional case, when signs of entries of the matrix Γ, and therefore of the matrix K, may have different signs, which is determined by complicate substitution-complimentary correlative relationships. It means that in multidimensional cases we faced the problem of existence of maximum of Total Revenue function, and that, in certain cases, the maximum may not exist at all. In this case one should be satisfied with OPPO model, and estimate optimal parameters of all m goods as independent values (Best, 2017).

Now we should clarify a criterion to detect whether the MPPO problem has the maximum point in the certain case. Such criterion can be constructed on the base of the Hessian matrix, which is the matrix of the second derivatives of TR_M quadratic forms with respect to prices. Let's denote the Hessian matrix as *H*, whereas the matrix of the first derivatives as - $K^{(1)}$. The m×1 vector of the first derivative of quadratic form (13) will be

$$TR_M^{(1)} = K^{(1)}p + b \quad (15)$$

where the entries of the matrix $K^{(1)}$ are

$$k_{ij}^{(1)} = \begin{cases} 2\gamma_{ij} & if\ i = j \\ \gamma_{ij} + \gamma_{ji} & if\ i \neq j \end{cases} \quad \text{(i,j=1, 2,…,m).}$$

It follows from (15) that the Hessian coincides with $K^{(1)}$.

It is well known fact that if the matrix H is positively definite at the critical point then quadratic form (13) has strict minimum, if the matrix is negatively definite then quadratic form (13) has strict maximum (Best, 2017). The definitions are general, and directly cannot be used as a criterion to detect whether there is a maximum. With this in view the following property of Hessian matrix can be used. Because the Hessian matrix is symmetric matrix its eigenvalues are real numbers, which implies the following important properties of Hessian.

Suppose x_0 is a critical point for a function f and there are the eigenvalues λ_i of the Hessian matrix. Then (Best, 2017):

(a) If all of the eigenvalues $\lambda_i > 0$, (i=1, 2,…,m) then at point x_0 is a strict local minimum of f;

(b) If all the eigenvalues, $\lambda_i < 0$ (i=1, 2,...,m) then at point x_0 is a strict local maximum of f.

(c) If at least one eigenvalue is positive and at least one eigenvalue is negative then at point x_0 is a saddle point of f.

The property (b) can be used as reliable and simple criterion of existing the optimal solution of the MPPO problem. The criterion can be reformulated in our case: if all the eigenvalues of the Hessian of the objective quadratic form of Total Revenue, $\lambda_i < 0$ (i=1, 2,...,m) then the problem has strict maximum.

We must outline, that Absence of local maximum of TR_M function of the MPPO is internal potential property of Multi-Product case. It means that in this case the optimal parameters should be calculated on the base of the consideration of system (11) as a simingly independent multidimensiomal linear regrssions. The latter leads to some conceptual and computational issues which will be discussed later.

Conclusion

One-Product Prices Optimization (OPPO) can't be used for Multi-Product Prices Optimization (MPPO) problem, because of both prices and demands, being random variables, may exhibit different covariate relationships, determined by means of substitutional and complimentarity processes. The system of covariate relationships, determined by means of substitution and complimentarity processes, is represented as the system of multivariance regression equations. MPPO problem have been reduced to the maximization problem of the objective function of Total Revenue, which is a quadratic form. In multidimensional cases one faces the problem of absence of local maximum of Total Revenue function. The simple criterion of existing the optimal solution of the MPPO problem is suggested. The criterion is: if all the eigenvalues of the Hessian of the objective quadratic form of Total Revenue, λ_i<0 (i=1, 2,...,m) then the problem has strict maximum. In the case of absence of the strict maximum of Total Revenue function, parameters should be calculated on the base of OPPO model, that is on the base of independent paired problems. The represented theory can be used as a basis for development of a relevant software.

References

Best, M. (2017). Quadratic programmng with computer programs. In C. a. Hall, *CRC Press Series: Avances in Applied Mathematics* (p. 386).

Gallego, G., & Wang, R. (2014). Multiproduct price optimization and competition under the nested logit model with product-differentiated price sensitivities. *Operations Research*, 450-461.

Goic, M. (2011). Essays on multi-product pricing. In Dissertations (p. 547). Carnegie Mellon University.

Soon, W., Zhao, G., & Zhang, J. (2009). Complementarity demand functions and pricing models for multi-product markets. *European Journal of Applied Mathematics, 20*(5), 399-430.